BINOMIAL THEOREM

An algebraic expression containing two terms is called a ***binomial expression*** *.e.g (a + b), (3x – 5y). Similarly an algebraic expression containing three terms is called* ***trinomial.*** *The general form of the binomial is (x + a) and the expansion of* $(x + a)^n$, $n \in N$ *is called* ***binomial theorem.***

Binomial Theorem (For a Positive Integral Index)

If n is a positive integer and x, a are two real or complex quantities, then

$$(x + a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n \ldots \text{(1)}$$

or

$$(x + a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

The coefficients ${}^nC_0, {}^nC_1 \ldots\ldots {}^nC_n$ are called binomial coefficients.

Properties of Binomial Expansion.

1. **Number of terms:** There are (n + 1) terms in the expansion of $(x + a)^n$, n being a positive integer.

2. In any term of expansion (1), the sum of the exponents of x and a is always constant = n.

3. The binomial coefficients of terms equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{n-r}$ $(0 \le r \le n)$.

4. **General Term:** The general term of the expansion is $(r+1)^{th}$ term usually denoted by

$$T_{r+1} = {}^nC_r\, x^{n-r} a^r \quad (0 \le r \le n).$$

5. **Middle Term:** The middle term in the expansion of $(x + a)^n$

   (a) If n is even then there is just one middle term, i.e., $\left(\frac{n}{2}+1\right)^{th} = {}^nC_{n/2}\, x^{n/2} a^{n/2}$

   (b) If n is odd, the there are two middle terms, i.e. $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ terms

6. To find the term independent of x or absolute or constant term in the expansion of $(x + a)^n$. Let T_{r+1} be the term independent of x. Equate to zero the index of x and you will find the value of r.
   Note: Term independent of x in Binomial Expansion means the co-efficient of x^0 in the expansion.

Ex. Find the middle term in the expansion of $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$

Sol. Here n = 20, which is an even number. So, $\left(\frac{20}{2}+1\right)^{th}$ term i.e. 11^{th} term is the middle term.

Hence, the middle term = $T_{11} = T_{10+1} = {}^{20}C_{10}\left(\frac{2}{3}x^2\right)^{20-10}\left(-\frac{3}{2x}\right)^{10} = {}^{20}C_{10}\ x^{10}$

7. **(m + 1) th Term From The End**:- To find (m + 1) th term from the end in the Binomial expansion of $(x + y)^n$ The (m + 1) th term from the end = (n – m + 1) th term from the beginning = T_{n-m+1} or find (m + 1) th term from the beginning in the Binomial expansion of $(y + x)^n$

Ex. Find the 11th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$

Sol. Clearly the given expansion contains 26 terms So, 11th term from the end = (26 – 11 + 1) th term from the beginning i.e. 16th term from the beginning.

Now, $T_{16} = T_{15+1} = {}^{25}C_{15}(2x)^{25-15}\left(-\frac{1}{x^2}\right)^{15} = {}^{25}C_{15} \cdot 2^{10} \cdot x^{10}\,\frac{(-1)^{15}}{x^{30}} = -\,{}^{25}C_{15} \cdot \frac{2^{10}}{x^{20}}$

Or T_{11} from end in $\left(2x - \frac{1}{x^2}\right)^{25}$ will be same as T_{11} from beginning in $\left(-\frac{1}{x^2} + 2x\right)^{25}$

Ex. Find the 10th term in the binomial expansion of $\left(2x^2 + \frac{1}{x}\right)^{12}$

Sol. $T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$. Therefore In the expansion of $(2x^2 + 1/x)^{12}$
we have $T_{10} = T_{9+1} = {}^{12}C_9\, (2x^2)^{12-9}\, (1/x)^9$
[Here : n = 12, r = 9, x = $2x^2$ and a = 1/x] = ${}^{12}C_9\, (2x^2)^3\, 1/x^9$
$= {}^{12}C_9\; 2^3\, (1/x^3) = {}^{12}C_3\, \frac{8}{x^3} = \frac{12x11x10}{3x2x1} x \frac{8}{x^3} = \frac{1760}{x^3}$ [Because ${}^{12}C_9 = {}^{12}C_3$]

8. **Coefficient Of x^m:** To find the coefficient of x^m $(m \le n)$ follow the following steps:-
   - **Step 1** First find $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$
   - **Step 2** Put the exponent of x = m if y is independent of x, i.e. n – r = m, or r = n – m. If y is also a function of x, simplify and get net power of x and equate it to m and find r.
   - **Step 3** If r = positive integer, or 0, find the coefficient of x^m and if r is a negative integer or a fraction then the coefficient of $x^m = 0$ or you can say coefficient of x^m doesn't exist in this binomial expansion.

9. Greatest coefficient in the expansion of $(x + y)^n = {}^nC_{n/2}$, if n is even.

MULTINOMIAL THEOREM

Another form of Binomial Expansion: We have, $(x+a)^n = \sum_{s=0}^{n} {}^nC_s\, x^{n-s} a^s = \sum_{s=0}^{n} \frac{n!}{(n-s)!\,s!} x^{n-s} a^s$

Putting n – s = r i.e. r + s = n, we get

$$(x+a)^n = \sum_{r+s=n} \frac{n!}{r!\,s!} x^r a^s$$

The general term in the above expansion is $\frac{n!}{r!\,s!} x^r a^s$.

The total number of terms in this expansion is equal to the number of non-negative integral solutions of r + s = n i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n+1$. Because, number of non-negative integral solutions of

$$x_1 + x_2 + \ldots\ldots + x_r = n \text{ is } {}^{n+r-1}C_{r-1}.$$

Multinomial Theorem: As discussed above, for any $n \in N$ we have,

$$(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1! r_2!} x_1^{r_1} x_2^{r_2}$$

This result can be generalized to the following form : $(x_1 + x_2 + x_3 + \ldots\ldots + x_k)^n$

$$= \sum \frac{n!}{r_1! r_2! r_3! \ldots r_k!} \times x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k} \text{ where } r_1 + r_2 + \ldots + r_k = n$$

The general term in the above expansion is

$$\frac{n!}{r_1! r_2! r_3! \ldots\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

The number of terms in the above expansion is equal to the number of non-negative integral solutions of the equation $r_1 + r_2 + \ldots\ldots r_k = n$, because each solution of this equation provides a term in above equation. The number of such solutions is ${}^{n+k-1}C_{k-1}$.

Note:

Total terms in the expansion of $(x_1 + x_2 + x_3 + \ldots\ldots + x_k)^n$ is equal to ${}^{n+k-1}C_{k-1}$.

e.g

- Number of terms in $(x_1 + x_2)^n = {}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$
- Number of terms in $(x_1 + x_2 + x_3)^n = {}^{n+3-1}C_{3-1} = {}^{n+2}C_2 = \frac{(n+1)(n+2)}{2}$

Some Important Results:

In solving problems relating the coefficients in the binomial expansion, we generally use the following results :

(I) Coefficient of $(r + 1)^{th}$ term in the binomial expression of $(1 + x)^n$ is ${}^nC_r$.

(II) Coefficient of x^r in the binomial expansion of $(1 + x)^n$ is ${}^nC_r$.

(III) Coefficient of x^r in the expansion of $(1 - x)^n$ is $(-1)^r\ {}^nC_r$.

(IV) Coefficient of $(r + 1)^{th}$ term in the expansion of $(1 - x)^n$ is $(-1)^r\ {}^nC_r$.

IMPORTANT CASES OF BINOMIAL EXPANSION

(1) If we put x = 1 in (1), we get

$$(1 + a)^n = {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + \ldots + {}^nC_r a^r + \ldots + {}^nC_n a^n \ \ldots\ldots\ldots\ (2)$$

(2) If we put x = 1 and replace a by – a, we get

$$(1 - a)^n = {}^nC_0 - {}^nC_1 a + {}^nC_2 a^2 - \ldots + (-1)^r\ {}^nC_r a^r + \ldots + (-1)^n\ {}^nC_n a^n \ \ldots\ (3)$$

(3) Adding and subtracting (2) and (3), and then dividing by 2, we get

$$\tfrac{1}{2}\{(1 + a)^n + (1 - a)^n\} = {}^nC_0 + {}^nC_2 a^2 + {}^nC_4 a^4 + \ldots\ (4)$$

$$\tfrac{1}{2}\{(1 + a)^n - (1 - a)^n\} = {}^nC_1 + {}^nC_3 a^3 + {}^nC_5 a^5 + \ldots\ (5)$$

Some Important Deduction:-If $C_0, C_1, C_2 \ldots C_{n-1}, C_n$ denote the binomial coefficients in the expansion of $(1 + a)^n$, then

(i) $C_0 + C_1 + C_2 + C_3 + \ldots\ldots C_n = 2^n$

(ii) $C_0 + C_2 + C_4 + C_6 + \ldots\ldots C_n = 2^{n-1}$

(iii) $C_1 + C_3 + C_5 + C_7 + \ldots\ldots C_{n-1} = 2^{n-1}$

(iv) $\frac{C_1}{C_0} + 2.\frac{C_2}{C_1} + 3.\frac{C_3}{C_2} + \ldots + n.\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

(v) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)(C_3 + C_4) \ldots (C_{n-1} + C_n) = \frac{C_0 C_1 C_2 \ldots C_{n-1}(n+1)^n}{n!}$

(vi) $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \ldots + C_{n-r} C_n = \frac{(2n)!}{(n-r)!(n+r)!}$

(vii) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = 2^n \frac{1.3.5\ldots\ldots(2n-1)}{n!}$

(viii) $C_0^2 - C_1^2 + C_2^2 + \ldots\ldots(-1)^n C_n^2 = \begin{cases} 0, \text{if } n \text{ is odd} \\ (-1)^{n/2}\ {}^nC_{n/2} \text{ if } n \text{ is even} \end{cases}$

Sol. (i) Put a = 1 in equation (2)

we get $(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_r + \ldots + {}^nC_n$ …… (2)

So $C_0 + C_1 + C_2 + C_3 + \ldots\ldots C_n = 2^n$

Note: The sum of the coefficient in a given binomial (or multinomial) expansion is obtained by replacing each variable by one

(ii) Put a = 1 in equation (4) and get $C_0 + C_2 + C_4 + C_6 + \ldots\ldots C_n = 2^{n-1}$

(iii) Put a = 1 in equation (5) and get $C_1 + C_3 + C_5 + C_7 + \ldots\ldots C_{n-1} = 2^{n-1}$

(iv) We have : $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{(n-r)!r!} \text{x} \frac{(n-r+1)!(r-1)!}{n!} = \frac{(n-r+1)}{r}, r = 1,2,3,\ldots n,$

Therefore $\frac{{}^nC_1}{{}^nC_0} = \frac{n}{1}, \frac{{}^nC_2}{{}^nC_1} = \frac{n-1}{2}, \frac{{}^nC_3}{{}^nC_2} = \frac{n-2}{3}, \ldots\ldots, \frac{{}^nC_n}{{}^nC_{n-1}} = \frac{1}{n}$

→ $\frac{{}^nC_1}{{}^nC_0} = \frac{n}{1}, \frac{{}^nC_2}{{}^nC_1} = \frac{n-1}{2}, \frac{{}^nC_3}{{}^nC_2} = \frac{n-2}{3}, \ldots\ldots, \frac{{}^nC_n}{{}^nC_{n-1}} = \frac{1}{n}$

Therefore $\frac{C_1}{C_0} + 2.\frac{C_2}{C_1} + 3.\frac{C_3}{C_2} + \ldots + n.\frac{C_n}{C_{n+1}} = \frac{n}{1} + 2.\left(\frac{n-1}{2}\right) + 3\left(\frac{n-2}{3}\right) + \ldots + n.\left(\frac{1}{n}\right)$

$= n + (n - 1) + (n - 2) + \ldots + 3 + 2 + 1 = \frac{n(n+1)}{2}$

(v) We have to prove that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)(C_3 + C_4) \ldots (C_{n-1} + C_n) = \frac{C_0 C_1 C_2 \ldots C_{n-1}}{n!}(n + 1)^n$

or $\left(\frac{C_0 + C_1}{C_0}\right).\left(\frac{C_1 + C_2}{C_1}\right).\left(\frac{C_2 + C_3}{C_2}\right)\ldots\left(\frac{C_{n-1} + C_n}{C_{n-1}}\right) = \frac{(n+1)^n}{n!}$

Now, $= \left(1 + \frac{C_1}{C_0}\right).\left(1 + \frac{C_2}{C_1}\right).\left(1 + \frac{C_3}{C2}\right)\ldots\left(1 + \frac{C_n}{C_{n-1}}\right)$

$= \left(1 + \frac{n}{1}\right).\left(1 + \frac{n-1}{2}\right).\left(1 + \frac{n-2}{3}\right).\left(1 + \frac{n-3}{4}\right)\ldots\left(1 + \frac{1}{n}\right)\left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}, r = 1,2\ldots,n\right]$

$= .\left(\frac{n+1}{1}\right).\left(\frac{n+1}{2}\right).\left(\frac{n+1}{3}\right)\left(\frac{n+1}{4}\right)\ldots\left(\frac{n+1}{n}\right) = \frac{(n+1)^n}{n!}$

(vi) Using binomial expansion, we have

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ …..(A) and

$(x + 1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_{n-1} x + C_n$ ….(B).

Multiplying (A) and (B) we get

$(1 + x)^{2n} = (C_0 + C_1 x + C_2 x^2 + \ldots + C_r x^r + \ldots + C_n x^n) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_r x^{n-r} + \ldots + C_{n-1}x + C_n)$

Equating the coefficients of x^{n-r} on both sides of (C), we get

$$C_0 C_r + C_1 C_{r+1} + \ldots + C_{n-r} C_n = {}^{2n}C_{n-r} = \frac{2n!}{(n-r)!(n+r)!}$$

GREATEST TERM IN THE EXPANSION OF $(x + a)^n$

Let T_{r+1} and T_r be $(r + 1)^{th}$ and r^{th} terms respectively in the expansion of $(x + a)^n$. Then,

$$T_{r+1} = {}^nC_r\, x^{n-r} a^r \text{ and } T_r = {}^nC_{r-1}\, x^{n-r+1} a^{r-1}$$

Therefore $\frac{T_{r+1}}{Tr} = \frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}} = \frac{n-r+1}{r}\frac{a}{x}$

Hence, $T_{r+1} \geq T_r \quad \frac{T_{r+1}}{T_r} \geq 1 \quad \frac{(n-r+1)a}{rx} \geq 1$

Find the optimum value of r from here.

Algorithm for finding the greatest term

STEP I : Write T_{r+1} and T_r from the given expansion.

STEP II : Find $\frac{T_{r+1}}{T_r}$

STEP III : Put $\frac{T_{r+1}}{T_r} > 1$

STEP IV : Solve the inequality in step III for r to get an inequality of the form $r < m$ or $r > m$.
If m is an integer, then mth and (m +1)th terms are equal in magnitude and these two are the greatest terms If m is not an integer, then obtain the integral part of m, say k. In this case, (k +1)th term is the greatest term.

Ex. Find the greatest term in the expansion of $(1 + x)^{10}$ when $x = 2/3$.

Sol. Let T and T_{r+1} denote the rth and (r + 1) th terms in the expansion of $(1 + x)^{10}$

Then $T_r = {}^{10}C_{r-1}$ and $T_{r+1} = {}^{10}C_r\, x^r$

Therefore $\frac{T_{r+1}}{T_r} = \frac{{}^{10}C_r x^r}{{}^{10}C_{r-1} x^{r-1}} = \frac{{}^{10}C_r}{{}^{10}C_{r-1}} x = \frac{10!}{(10-r)!r!} x \frac{(10-r+1)!(r-1)!}{10!} x$

$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{11-r}{r} x \Rightarrow \frac{T_{r+1}}{T_r} = \left(\frac{11-r}{r}\right) x \frac{2}{3} \left[\because x = \frac{2}{3}\right]$

Now, $\frac{T_{r+1}}{T_r} > 1 \Rightarrow \left(\frac{11-r}{r}\right) x \frac{2}{3} > 1 \Rightarrow 22 > 5r \Rightarrow r < 4\frac{2}{5}$

Therefore (4 + 1)th, ie., 5^{th} term is the greatest term. Putting r = 4 in T_{r+1} we get $T_5 = {}^{10}C_4\, x^4$

$\Rightarrow T_5 = {}^{10}C_4 \left(\frac{2}{3}\right)^4 \left[\because x = \frac{2}{3}\right] \Rightarrow T_5 = 210 \left(\frac{2}{3}\right)^4$

BINOMIAL THEOREM FOR ANY INDEX

Statement. Let n be a rational number and x be a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots\infty$$

Remark:

1. The condition $|x| < 1$ is un-necessary, if n is a whole number while the same condition is essential if n is a rational number other than a whole number.
2. Note that there are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.
3. In the above expansion the first term is unity. If the first term is not unity and the index of the binomial is either a negative integer or a fraction, then we expand as follows:

$$(x+a)^n = \left\{a\left[1+\frac{x}{a}\right]\right\}^n = a^n\left[1+\frac{x}{a}\right]^n = a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\ldots\ldots\ldots\ldots\right\}$$

$= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots$ This expansion is valid when $\left|\frac{x}{a}\right| < 1$ or equivalently $|x| < |a|$.

4. If n is a positive integer the above expansion contains(n + 1) terms and coincides with

$$\mathbf{(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n,}$$

because ${}^nC_0 = 1$, ${}^nC_1 = n$, ${}^nC_2 = n\frac{(n-1)}{2!}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$,......

General Term in the Expansion of $(1+x)^n$ The general term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{1.2.3.4\ldots\ldots\ldots\ldots r}x^r$$

SOME IMPORTANT DEDUCTIONS

1. Replacing n by –n in the expansion for $(1+x)^n$, we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}x^r + \ldots\ldots$$

The general term in this expansion is $T_{r+1} = (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r$

2. Replacing x by – x and n by –n in the expansion of $(1+x)^n$ we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}x^r + \ldots$$

The general term in this expansion is $T_{r+1} = \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r$

3. Replacing x by – x in the expansion of $(1+x)^n$, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

The general terms is $T_{r+1} = \frac{(-1)^r n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$

Ex. Find the general term in the expansion of $(2-3x^2)^{-2/3}$

Sol. We have $(2-3x^2)^{-2/3} = 2^{-2/3} = 2^{-2/3}\left(1-\frac{3x^2}{2}\right)^{-2/3}$. Let T_{r+1} be the $(r+1)^{th}$ term the binomial expansion of $\left(1-\frac{3x^2}{2}\right)^{-2/3}$ Then $T_{r+1} = \frac{\left(\frac{-2}{3}\right)\left(\frac{-2}{3}-1\right)\left(\frac{-2}{3}-2\right)....\left(\frac{-2}{3}-(r-1)\right)}{r!}\left(\frac{-3}{2}x^2\right)^r$

$= (-1)^r \frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right).....\left(\frac{3r-1}{3}\right)}{r!} (-1)^r . \frac{3^r}{2^r} x2^r = (-1)^{2r} \frac{2.5.8...(3r-1)}{r!3^r}\frac{3^r}{2^r} = x^{2r} = \frac{2.5.8.....(3r-1)}{r!2^r} x^{2r}$

Hence, the required general term = $2^{-2/3} \frac{2.5.8.....(3r-1)}{r!2^r} x^{2r}$

Keep in mind

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \infty$
(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\infty$
(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\infty$
(iv) $(1-x)^{2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$
(v) $(1+x)^{-3} = 1 - 3x + 6x^2 \ldots$
(vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + \ldots$

Question of type $(x + \sqrt{y})^n$

Algorithm to solve this problems:

STEP I: Write the given expression equal to I + F, where I is its integral part and F is the fractional part.

STEP II: Define G by replacing '+' sign in the given expression by '–'. Note that G always lies between 0 and 1.

STEP III: Either add G to the expression in step I or subtract G from the expression in step I so that RHS is an integer.

STEP IV: If G is added to the expression in step I, then G + F will always come out to the equal to 1 i.e. G = 1 – F. If
G is subtracted from the expression in step I, then G will always come out to be equal to F.

STEP V: Obtain the value of the desired expression after getting F in terms of G.
Following examples illustrate the above procedure.

Ex. If $(5 + 2\sqrt{6})^n = I + f$, where I and n are positive integers and f is a positive fraction less than one, show that $(I + f)(1 - f) = 1$.

Sol. Clearly, I and f are respectively the integral and fractional parts of $(5 + 2\sqrt{6})^n$.
Let $G = (5 - 2\sqrt{6})^n$. Since $0 < 5 - 2\sqrt{6} < 1$, therefore $0 < G < 1$.
Now $I + f + G = (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n = 2[{}^nC_0\, 5^n + {}^nC_2\, 5^{n-2}(2\sqrt{6})^2 + {}^nC_4\, 5^{n-4}(2\sqrt{6})^4 + \ldots]$
= an even integer λ (say)

$\Rightarrow f + G = \lambda - 1 \quad \Rightarrow f + G$ is an integer $\quad \left[\begin{array}{l}\because \lambda \in Z, I \in Z \\ \Rightarrow \lambda - I \in Z\end{array}\right]$

$\Rightarrow f + G = 1$ $[\therefore 0 < f < 1, 0 < G < 1 \quad \therefore 0 < f + G < 2 \Rightarrow f + G$ is an integer between 0 and 2 $\Rightarrow f + G = 1]$

$\Rightarrow G = 1 - f$

Now, $(I + f)(1 - f) = (I + f) G = (5 + 2\sqrt{6})^n (5 - 2\sqrt{6})^n = 1^n = 1$.

Ex. If $(7 + 4\sqrt{3})^n = I + F$, where I is a positive integer and F is a proper fraction, show that $(I + F)(1 - F) = 1$.

Sol. Let $G = (7 - 4\sqrt{3})^n$.

Clearly, if we add G and I + F, we get an integer i.e.

$I + F + G = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2\,({}^nC_0\, 7^n + {}^nC_2\, 7^{n-2} (4\sqrt{3})^2 + \ldots)$ = an even integer.

$\therefore F + G = 1 \Rightarrow G = 1 - F$

Hence, $(I + F)(1 - F) = (I + F) G = (7 + 4\sqrt{3})^n (7 - 4\sqrt{3})^n = 1$

EXAMPLES

1. Write the general term in the expansion of $(x^2 - y)^6$

Sol. We have $(x^2 - y)^6 = (x^2 + (-y))^6$ The general term in the expansion of the above binomial is given by

$T_{r+1} = {}^6C_r (x^2)^{6-r} (-y)^r$ [Because $T_{r+1} = {}^nC_r\, x^{n-r} a^r$] or $T_{r+1} = (-1)^r\, {}^6C_r\, x^{12-2r} y^r$

2. Find the coefficients of x^{32} and x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Sol. Suppose (r + 1) th term involves x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Now, $T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r = (-1)^r\, {}^{15}C_r\, x^{60-7r}$ ....(1)

For this term contain x^{32}, we have $60 - 7r = 32 \Rightarrow r = 4$. So, (4 + 1)th i.e. 5th term contains x^{32}

Putting r = 4 in (i), we get $T_5 = (-1)^4\, {}^{15}C_4\, x^{(60-28)} = {}^{15}C_4\, x^{32}$ Therefore Coefficient of $x^{32} = {}^{15}C_4$

= 1365. Suppose (s + 1) th term in the binomial expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ contains x^{-17} Now,

$T_{s+1} = {}^{15}C_s (x^4)^{15-s} \left(-\frac{1}{x^3}\right)^s = (-1)^s\, {}^{15}C_s\, x^{60-7s}$ ....(ii) If this term contains x^{-17}, we have

$60 - 7s = -17 \Rightarrow s = 11$. So, (11 + 1)th i.e. 12th term contains x^{-17} .Putting s = 11 in (ii), we get

$T_{12} = (-1)^{11}\, {}^{15}C_{11}\, x^{-17} = -{}^{15}C_{11}\, x^{-17} = -{}^{15}C_4\, x^{-17}$ [Because ${}^nC_r = {}^nC_{n-r}$].

Therefore Coefficient of $x^{-17} = -{}^{15}C_4 = -1365$.

3. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is $\left[\frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}\right]$

Sol. Suppose x^p occurs in (r + 1) th term in the expansion of $(x^2 + 1/x)^{2n}$

Now $T_{r+1} = {}^{2n}C_r (x^2)^{2n-r} \left(\frac{1}{x}\right)^r = {}^{2n}C_r\, x^{4n-3r}$ .... (i). For this term to contain x^p , we must have

$4n - 3r = p \Rightarrow r = \frac{4n-p}{3}$. Therefore Coefficient of $x^p = {}^{2n}C_r$, where $r = \frac{4n-p}{3}$

$$= \frac{(2n)!}{(2n-r)!r!} \text{ where } r = \left(\frac{4n-p}{3}\right) = \frac{(2n)!}{\left\{2n-\left(\frac{4n-p}{3}\right)\right\}!\left(\frac{4n-p}{3}\right)!} = \frac{(2n)!}{\left(\frac{2n+p}{3}\right)!\left(\frac{4n-p}{3}\right)!}$$

4 Find the coefficient of x^{40} in the expansion of $(1 + 2x + x^2)^{27}$

Sol. We have : $(1 + 2x + x^2)^{27} = \{(1 + x)^2\}^{27} = (1 + x)^{54}$ Suppose x^{40} occurs in (r + 1) th term in the expansion of $(1 + x)^{54}$. Now $T_{r+1} = {}^{54}C_r\, x^r$. For this term to contain x^{40} we must have r = 40. So, coefficient of $x^{40} = {}^{54}C_{40}$

5. If the 6^{th} term in the expansion of $\left(\frac{1}{x^{8/3}} + x^2 \log_{10} x\right)^8$ is 5600, find the value of x.

Sol. We have $T_6 = 5600 \Rightarrow T_{5+1} = 5600$

$$\Rightarrow \quad {}^8C_5 \left(\frac{1}{x^{8/3}}\right)^{8-5} (x^2 \log_{10} x)^5 = 5600 \Rightarrow 56x^2 (\log_{10}x)^5 = 5600$$

$$\Rightarrow \quad x^2 (\log_{10} x)^5 = 100 \Rightarrow x^2(\log_{10} x)^5 = 10^2 \Rightarrow x^2 (\log_{10} x)^5 = 10^2 (\log_{10} 10)^5 \Rightarrow x = 10.$$

6. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7: 42. Find n.

Sol. Let the three consecutive terms be r^{th} $(r + 1)^{th}$ and $(r + 2)^{th}$ terms. Their coefficients in the expansion of $(1 + x)^n$ are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively it is given that

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42 \text{ . Now , } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \Rightarrow \frac{n!}{(n-r+1)!(r-1)!} \times \frac{(n-r)!r!}{n!} = \frac{1}{7}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{1}{7} \Rightarrow n - 8r + 1 = 0 \text{ ..... (i) and,}$$

$$= \frac{7}{42} \Rightarrow \frac{n!}{(n-r)!r!} \times \frac{(n-r-1)!(r+1)!}{n!} \frac{1}{6} \Rightarrow \frac{r+1}{n-r} = \frac{1}{6} \Rightarrow n - 7r - 6 = 0 \text{ .....(ii)}$$

Solving (i) and (ii), we get r = 7 and n = 55.